##  UNIT- III

The general solution of the differential equation

Where and is

Where, the complementary function, is the general solution of the related homogeneous equation of (1) and is the particular integral of (1)

Here can only contain terms such as and a finite number of combinations of such terms.

To find by this method, it is necessary to compare two terms in (1) with those of . In this process of comparison different possibilities occur. We shall consider each of these and explain then with the help of a working rule and solved examples.

**Case (i):-** **working rule**

If no term of in equation (1) is same as a term in , then will be a linear combination of the terms of and all its linearly independent derivatives.

**Table**

|  |  |  |
| --- | --- | --- |
| **S.No** | **Special form of**  | **Trail solution for P.I** |
| 1. |  or or  |  |
| 2. |  or  |  |
| 3. |  or  |  |
| 4. |  or or  |  |
| 5. |  or or  |  |
| 6. |  (or) (or) (or) (or) (or)  |  |

**Solved problems**

**Ex(1):-** Solve

**Sol:-** Given equation is

 Auxiliary equation is

 Since has no term common with will be a linear combination of and all its linearly independent derivatives

 Since is a solution of equation (1)

 Substitute in equation (1), we get

 Compare the coefficients of & constants on both sides

 The general solution of (1) is

**Ex(2):-** Solve

**Sol:-** Given

 Auxiliary equation is

 Since has no term common with . Hence will be a linear combination of and all its L.I derivatives. Neglecting the constant coefficients

 ( are to be determined)

 Substituting the values of in equation (1), we get

 Equating the coefficients of like terms on both sides

**Ex(3):-** Solve

**Sol:-** Given

 Auxiliary equation is

 Since has no term common with . Hence will be a linear combination of and all its linear integration derivatives

 Substituting above values in equation (1), we get

 Compare the coefficients of on both sides we get

 The general solution of (1) is

**Ex(4):-** Solve

**Sol:-** Given

 Auxiliary equation is

 Here has no term common with hence will be a linear combination of and all its linear integration derivatives

 Substituting these values in equation (1)

 Compare the coefficients of and on both sides

**Ex(5):-** Solve

**Sol:-** Given

 Auxiliary equation is

 Here has no term common with . Hence will be a linear combination of and all its linear integration derivatives

Substitute the values in (1), we get

The general solution of (1) is

**Ex(6):-** Solve

**Sol:-** Given

 Auxiliary equation is

 Hence has no term common with. Hence will be a linear combination of and all its linearly independent derivatives

Since is a solution of (1) substituting the values in (1) weget

 The general solution of (1) is

**Exercise**

The following differential equations by the method of undetermined coefficients

(1)

(2)

**Answers**

(1)

(2)

**Case (ii):-** If in equation (1) contains a term which is terms a term of where is zero or a positive integer. The particular integral of (1) will be a linear combination of and all its linearly independent derivatives. Also if in addition, contains terms which correspond to case (i). Then proper terms required by this must also be included in

**Solved problems**

**Ex(1):-** Solve

**Sol:-** Given equation is

 Auxiliary equation is

 Here

 Compare (3) with (2). We observe that contains which is terms of the same term in . Hence for this term, must contain a linear combination of and all its linear independent derivatives.

Also has the term which is not present in . Hence by case (i), must include a linear combination of it and all its linear integration derivatives ignore the constant coefficients we can neglect as it already appeared in

Substitute the above value in (1), we get

 Equating the coefficient of like terms, on both sides

 The general solution of (1) is

**Ex(2):-** Solve

**Sol:-** Given differential equation is

 Auxiliary equation is

 Here comparing this with we observe contains which is times the same term in . Hence for this term must contain a linear combination of and all its linear independent derivatives. Also contains the term for this we include a linear combination of it and its derivatives

Substitute the above values in (1), we get

 Equating the coefficient of like terms, on both sides

 The general solution of (1) is

**Ex(3):-** Solve

**Sol:-** Given

 Auxiliary equation is

 Here

Compare (2) & (3): we observe that contain a term which is times of the same term in. Hence for this term must contain a linear combination of and its linear derivatives.

Substitute the above values in (1), we get

Compare the coefficients of like terms on both sides

The general solution of (1) is

**Exercise**

Solve the following differential equations by the method of undetermined coefficients

(1)

(2)

(3)

**Answers**

(1)

(2)

(3)

**Case (iii):-** The general solution of the differential equation

Where and is

(i) The auxiliary equation of (1) has a multiple root and (ii) contains a term (neglecting the constant coefficients) Where is a term in and is obtained from the multiple roots, then will be a linear combination of and all its linearly independent derivatives. In addition if contains terms as in the case (i) and (ii), then the proper terms must also be included according in .

**Solved problems**

**Ex(1):-** Solve

**Sol:-** Given

 Auxiliary equation is

 The auxiliary equation has a multiple roots . Contains the term which is timesthe term in and this term in came from a multiple root. Hence, and

 must be a linear combination of and all its L.I derivatives. In obtaining this linear combination, we neglect and as they are already in

Substituting the values of in (1), we get

Comparing coefficients of like terms, we have

The general solution of (1) is

**Ex(2):-** Solve

**Sol:-** Given

 Auxiliary equation is

 The auxiliary equation has a multiple roots . contains the term which is times the term in and this term in came from a multiple root. Hence, and

 must be a linear combination of and all its L.I derivatives. In obtaining this linear combination, we neglect and as they are already in

Substituting the values of in (1), we get

Comparing coefficients of like terms on both sides

The general solution of (1) is

**Ex(3):-** Solve

**Sol:-** Given

 Auxiliary equation is

 (Repeated roots)

 The auxiliary equation has a multiple root .

 Contain the term which is times the term in and this term came from a multiple root. Hence , and

 must be a linear combination of and all its L.I derivatives.

 And

Substituting the values of in (1), we get

 The general solution is

**Exercise**

(1)

(2)

(3)

**Answers**

(1)

 (2)

(3)

 Unit-IV

**Method of variation of parameters**

General solution of by the method of variation of parameters.

Given linear differential equation is where and are functions of or real constants and is only a function of, its homogeneous equation corresponding to (1) is

Let be the general solution of (2)

Where are functions of and are real constants hence it is the complementary function of (1)

 Satisfies (2)

And

Let a particular integral of (1) be by replacing and by and respectively. Which are also some functions of

Choose and such that

Then

And

Substituting these values in (1), we get

 Using (3) and (4)

Solving (6) and (8)

 and

 And

After integration the constant is not added since and are involved in.

Substituting the values of and

From (9) in (5), we get

 The general solution of (1) is

**Another method:-**

General solution of by the method of variation of parameters.

**Proof:-** Given linear differential equation is where are fuctions of or real constants and is only a function of . The homogeneous equation corresponding to (1) is

Let be the general solution of (2) where and are solutions of (2) and are real constants. Hence it is the complementary function of (1)

Let P.I of (1) is

Which is obtained from complementary function of (1) by replacing and by and respectively which are also come functions of and whose values are to be determined.

Differentiating equation (8) twice, we get

As is a solution of (1)

We have

Substituting (3),(4) and (5) in (6)

We have , and

(7) and (8)

Choose and such that

(9) & (10)

Solving (10) and (11)

Integrating the equations in (12), we can find and , substitute these and in (3)

We get of (1)

Hence the general solution of (1) is

**Solved problems**

**Ex(1):-** Solve

**Sol:-** Given

 Auxiliary equation is

 Let

 Let where to be determined

Where

 The general solution is

**Ex(2):-** Solve

**Sol:-** Given

 Auxiliary equation is

 ,

 Let

 Let

 The general solution is

**Ex(3):-** Find the particular integral of by using the method of variation of parameters.

**Sol:-** The complementary function of the given equation is

 Let

 Now

 Let be the particular integral of the given equation

**Ex(4):-** Solve

**Sol:-** Here

 Let

 Let

 Here

 Now

**Ex(5):-** Find the particular integral of

**Sol:-**

 Auxiliary equation is

 Let

 Let is a particular integral of the given equation

**Ex(6):-** Find the general solution of

**Sol:-**

 Auxiliary equation is

 Let

 Let

 Now

**Ex(7):-** Solve

**Sol:-**

 Auxiliary equation is

 Let

 Let

 General solution is

**Ex(8):-** Find the particular integral of by using method of variation of parameters

**Sol:-** Given

 Auxiliary equation is

 Let

 Let

 Now

**Ex(8):-** Find the particular integral of by using the method of variation of parameters.

**Sol:-** Given

 Auxiliary equation is

 Let

 Let

 Now

**Ex(9):-** Solve

**Sol:-** Given

 Auxiliary equation is

 Let

 Let

 Now

 The general solution of (1) is

**Ex(10):-** Solve using the method of variation of parameter

**Sol:-** Given

 Auxiliary equation is

 Let

 Let

**Ex(11):-** Solve using the method of variation of parameter

**Sol:-** Given

 Auxiliary equation is

 Let

 Let

**Exercise**

Solve the differential equations using the method of variation of parameters.

(1)

(2)

(3)

(4)

**Answers**

(1)

(2)

(3)

(4)

**Linear differential equations with non constant coefficients**

**Definition:-** An equation of the form where and are real valued functions of defined on an internal I, is called the linear equation of the second order with variable coefficients. If equation (1) is not solved by linear equations with constant coefficients, homogeneous equations and exact equations. We shall try the methods of this chapter.

Complete solution of in terms of one known integral belonging to complementary function. Solution of by reduction of its order

**Proof:-** Given

 Let be a known integral of the complementary function. So is a solution of (1) when its right hand side is taken to be zero. Thus is a solution of

 So that

 Now let the complete solution of (1) be

 Where is a function of . will now be determined

 From (3)

 Using (3) and (4), (1) becomes

 (from (2))

 Put so that

 Hence (5) gives

 Which is a linear equation in and .

The original equation has therefore had its order depresses by unity

 I.F

 And solution of (6) is

 Putting this value of in (3), we get

 Which includes the given solution and since it contains two arbitrary constants and . Hence it is the required complete solution.

**Theorem:** If is a solution of the equation, then is the general solution of the given equation, where being arbitrary constants

**Proof:** Given

 Given that is a solution of (1), hence

 Let

Be the second solution of (1). Hence, we have

From (3)

 ,

 Equation (4) reduces to

 (or)

 (or)

 using (2)

Let so that

Then, the above equation becomes

By variable & separable

Integrating it,

Where we have omitted the usual constant of integration because we wish to find second particular solution (1)

Thus

Which is the required form of the function

Now we shall show that and are linearly independent solutions of (1) we have

Wronskian of and

 , using (3)

 , on applying the operation

 , using (6)

Which is non-zero, being an exponential function. Since, if follows that and are linearly independent solution of (1)

Then, the general solution of (1) is

i.e

Rules for getting an integral belonging to complementary function (C.F) i.e solution of

**Rule 1:** is a solution of (1) if

**Proof:-** If then and

Putting these in (1), we get

 or

 Particular case (i) take . Then is a solution of (1) if

 Particular case (ii) take . Then is a solution of (1) if

**Rule 2:** is a solution of (1) if

**Proof:-** If , then and

 Putting these values in (1), we get

Particular case (i) Take , then is a solution of (1) if

Particular case (ii) Take , then is a solution of (1) if

Working rule for finding complete primitive solution when an integral of complementary function is known

**Step 1:** Put the given equation in standard form , in which the coefficients of is unity

**Step 2:** Find an integral of complementary function by using the following table

|  |  |
| --- | --- |
| **Condition satisfied** | **An integral of complementary function is** |
| (i) (ii) (iii) (iv) (v) (vi)  |  |

If a solution (or integral) is given in a problem, then this step is emitted

**Step 3:** Assume that the complete solution of given equation is , where has been obtained in step 2. Then given equation reduced to

**Step 4:** Take so that put in (1), Then (1) will come out to be a linear equation in and if . Solve it as usual if , then variables and will be separable.

**Step 5:** Now replace by and separate the variables and . Integrate and determine . Put this value of in the assumed solution . This will lead us to the desired complete solution of the given equation.

**Solved problems**

**Ex(1):-** Solve given as a solution.

**Sol:-** Given equation

 Also given is a solution (1)

 Let be the general solution (1)

 Where

 Here

 Substitute in (1), we get

 Put

 From (2)

 Variable & separable

 The general solution (1) is

**Ex(2):-** Given that is a solution of find the general solution of

**Sol:-** Given

 Also given is a solution of (1)

 Let be the general solution (1)

 Where

 Here

 Substitute in (1), we get

 Put

 From (2)

 Equation (3) is a linear equation in and

 I.F

 General solution of (3) is

 The general solution of (1) is

**Ex(3):-** Solve given that is a solution.

**Sol:-** Compare the given equation with

 Suppose is a solution of

 Then we have

 Now and

 is another solution of the given equation

 The general solution of the given equation is

**Ex(4):-** Solve given that is a solution.

**Sol:-** Compare the given equation with

 Suppose is a solution of

 Then we have

 Where

The general solution is

**Ex(5):-** Solve given that is a solution.

**Sol:-** Compare the given equation with

 Suppose is a solution of

 Now

 The general solution of given equation is

**Ex(6):-** Solve given that is a solution.

**Sol:-** Compare the given equation with

 Also here . Hence the second solution where

 Now

 The general solution of given equation is

**Ex(7):-** If is a solution of then solve it by the method of reduction of the order.

**Sol:-** Comparing the given equation with

 and

 Also here .

Hence the second solution

 Second solution is

 The general solution of given equation is

**Ex(8):-** Solve

**Sol:-** Given equation is

 Here

 Now

 is a part of complementary function of (1)

 Let

 Let be the general solution of (1)

 Put in (1) we get

 Let

 Equation (2) is a linear differential equation in (2)

 I.F

 The general solution of (2) is

 The general solution of (1) is

**Ex(9):-** Solve

**Sol:-** Given

 Here

 Now

 is a part of complementary function of given differential equation

 Let

 Put in (1), we get

 Put

 Equation (2) is a linear differential equation in and

 I.F

 The general solution of (2) is

 The general solution of (1) is

**Ex(10):-** Solve given is a solution.

**Sol:- Given**

 Here

 Now

 is a solution of (1)

 Let be the general solution of (1)

 Let

 Substituting in (1)

 Put

 The general solution of (1) is

**Ex(11):-** Given that is a solution of . Find the general solution of

**Sol:-** Given equation is

 Where

 Now

 is a solution of

 Let be the general solution of (1)

 Let

 Substitute in (1)

 General solution of (1) is

**Ex(12):-** Given that is a solution of . Find the general solution of

**Sol:-** Given equation is

 Given that is solution of

 Let be the general solution of (1)

 Let

 Substitute in (1) we get

 Put

 From (2)

 Equation (3) is a complementary differential equation in and

 I.F

 General solution of (3) is

 The general solution of (1) is

**Exercise**

Use the method of reduction of order to find the solutions of the following equations with a given solution

(1) given that is a solution.

(2) , given that is a solution.

(3) given that is a solution.

(4) Given that is a solution of . Find the general solution of

**Answers**

(1)

(2)

(3)

(4)

**The Cauchy-Euler equation:-**

**Homogeneous linear equations (or Cauchy- Euler equations):-**

A linear differential equation of the form

i.e

Where are constants and is either a constant or a function of only is called a homogeneous linear differential equation

**Method of solution of homogeneous linear differential equation**

In order to solve (1) introduce a new independent variable such that

 or so that

 Now using (2)

 (or)

 (by (2))

 And so on

Substituting the above values of in (1) and thus changing the independent variable from to . We have

 (or)

Where is now a function of only

**Working rule for solving linear homogeneous differential equation**

**Step(i):-** Put where

**Step(ii):-** Assume that and

 Then, we have

 and so on

 Then equation (1) reduced to

 Equation (2) is a linear differential equation with constant coefficients

The differential operator and the inverse operator obey the properties of and . Hence can be solved by the methods discussed already in this chapter

**Solved problems**

**Ex(1):-** Solve

**Sol:-** Given

 Put

 Equation (1) becomes

 Auxiliary equation is

 P.I

**Ex(2):-** Solve

**Sol:-** Given equation is

 Let

 Equation (1) becomes

 Auxiliary equation is

**Ex(3):-** Solve

**Sol:-** Given equation is

 Let

 Equation (1) becomes

 Auxiliary equation is

**Ex(4):-**

**Sol:-** Given

 Let

 Auxiliary equation is

**Exercise**

Solve the following Cauchy euler equation

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

**Answers**

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

**Legendre’s Equations**

A linear differential equation of the form

Where are constants and is either a constant or a function of only is called a Legendre’s linear equation

**Method of solution**

Let

Let and

From (2) we have

Again

 (frm (4))

 …………………….

 …………………….

Substituting the above values in (1), we get

Which is a linear differential equation with constant coefficients in variables and

 is now function of z only and is obtained by using transformation (2) by replacing by

Let a solution of (1) be

Then the required solution is given by

 as

**Solved problems**

**Ex(1):-** Solve

**Sol:-** Given

 Put

 Equation (1) becomes

 Auxiliary equation is

 The general solution of (3) is

 The general solution of (1) is

**Ex(2):-** Solve

**Sol:-** Given

 Put

 From (1)

 Auxiliary equation is

 The general solution of (2) is

 The general solution of (1) is

**Exercise**

(1) Solve

(2) solve

(3) solve

**Answers**

(1)

(2)

(3)